Stability of discrete shock profiles for systems of conservation laws

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- **Context and definition of discrete shock profiles**
- **Existence results**
- **Stability of discrete shock profiles**
    - Definition of the nonlinear orbital stability and overview of results
    - Main result : Spectral stability implies linear orbital stability

Conservation laws and shocks

We consider a one-dimensional scalar conservation law

$$\begin{gathered} \partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \end{gathered} \tag{1}$$

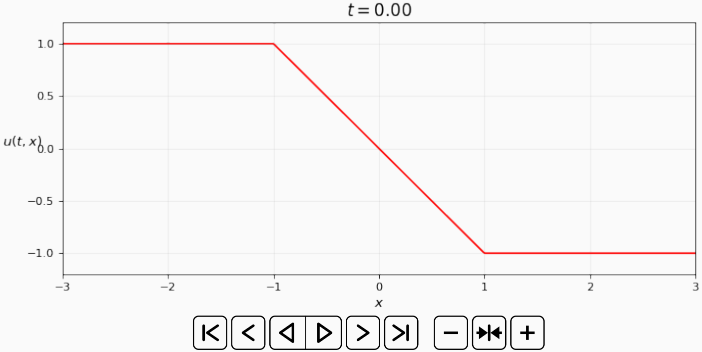
where the flux $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

The result that will be presented also holds for systems of conservations laws.

This type of PDE tends to have solutions with discontinuities.

We consider the Burgers equation $f(u) = \frac{u^2}{2}$ and thus $f'(u) = u$.

$$\partial_t u + u \partial_x u = 0$$



Larger goal: We want to know if numerical schemes obtained by discretizing our PDE can approach correctly the discontinuous solutions.

We consider two distinct states $u^-, u^+ \in \mathbb{R}^2$ and a speed $s \in \mathbb{R}$. The function u defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t,x) := \begin{cases} u^- & \text{if } x < st, \\ u^+ & \text{else,} \end{cases}$$

is a weak solution of the scalar conservation law if and only if

$$f(u^-) - f(u^+) = s(u^- - u^+). \quad \text{(Rankine-Hugoniot condition)}$$

It is a Lax shock when

$$f'(u^+) < s < f'(u^-).$$

The main result of the presentation will focus on steady Lax shocks, i.e. when $s = 0$.

Conservative finite difference schemes

We consider a conservative explicit finite difference scheme

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N} u^n$$

where for $u = (u_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $j \in \mathbb{Z}$ as

$$\mathcal{N}(u)_j := u_j - \nu \left(F\left(\nu; u_{j-p+1}, \ldots, u_{j+q}\right) - F\left(\nu; u_{j-p}, \ldots, u_{j+q-1}\right)\right).$$

- $u^0 \in \mathbb{R}^{\mathbb{Z}}$: initial condition
- $\mathcal{N} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$: nonlinear discrete evolution operator
- $F :]0, +\infty[\times \mathbb{R}^{p+q} \to \mathbb{R}$: numerical flux
- $p, q \in \mathbb{N} \setminus \{0\}$: integers defining the size of the stencil of the scheme.
- $\nu = \frac{\Delta t}{\Delta x} > 0$: ratio between the time and space steps.

The value u_j^n must be a good approximation of a solution u on $[n\Delta t, (n+1)\Delta t[\times \left[\left(j - \frac{1}{2}\right)\Delta x, \left(j + \frac{1}{2}\right)\Delta x\right[$

Assumptions:

- $\forall u \in \mathbb{R}, \quad F(\nu; u, \ldots, u) = f(u)$ (consistency condition)
- For some neighborhood $\mathcal{U}$ of the states $u^{\pm}$

$$\forall u \in \mathcal{U}, \quad -q \leq \nu f'(u) \leq p \quad \text{(CFL condition on } \nu\text{)}$$

- Linear-ℓ^2 stability for constant states $u \in \mathcal{U}$
- The scheme introduces numerical viscosity. In the present presentation, we consider a first order scheme. This excludes dispersive schemes like for instance the Lax-Wendroff scheme.

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$. For the numerical scheme, we consider the modified Lax Friedrichs scheme

$$\forall u \in \mathbb{R}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \quad \mathcal{N}(u)_j := \frac{u_{j+1} + u_j + u_{j-1}}{3} - \nu \frac{f(u_{j+1}) - f(u_{j-1})}{2}.$$

Discrete shock profiles

Discrete shock profile (DSP): A discrete shock profile is a solution of the numerical scheme of the form

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^n = \overline{u}(j - s\nu n)$$

where the function $\overline{u} : \mathbb{Z} + s\nu\mathbb{Z} \to \mathbb{R}$ verifies that

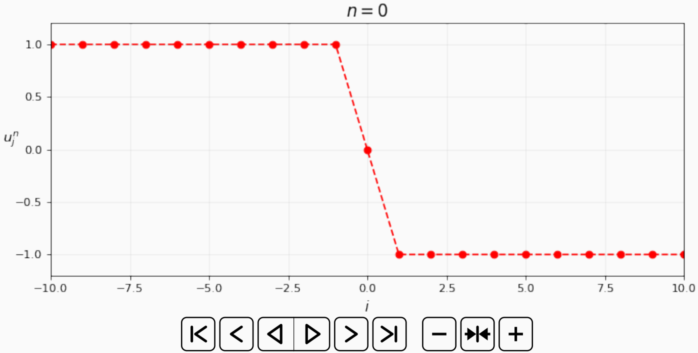
$$\overline{u}(x) \underset{x \to \pm\infty}{\to} u^{\pm}.$$

Stationary discrete shock profiles (SDSP) are sequences $\overline{u} = (\overline{u}_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfy

$$\mathcal{N}(\overline{u}) = \overline{u} \quad \text{and} \quad \overline{u}_j \underset{j \to \pm\infty}{\to} u^{\pm}.$$

Example : We consider the initial condition (mean of the standing shock on each cell $\left[(j-\frac{1}{2})\Delta x, (j+\frac{1}{2})\Delta x\right[$)

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$



Main goal: Finding conditions on the numerical schemes so that

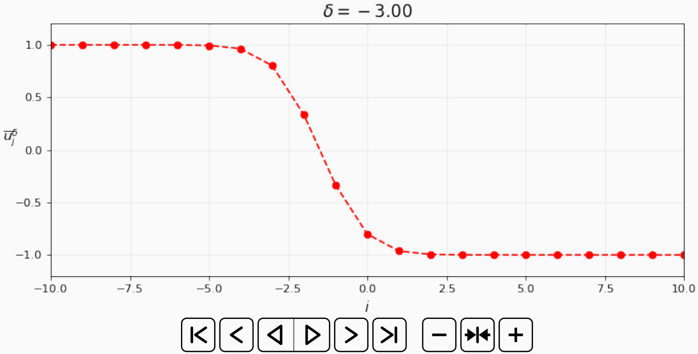
stable shock waves for the PDE $\Rightarrow$ stable DSPs for the numerical scheme

This separates the theory surrounding DSPs in two parts:

- Existence of DSPs
- Stability of DSPs

From now on we will focus on elements of theory surrounding stationary discrete shock profiles ($s=0$).

<u>**Example :**</u> We consider the same initial condition u^0 as before but add a mass δ at $j = 0$. We look at the limit of the solution of the numerical scheme.



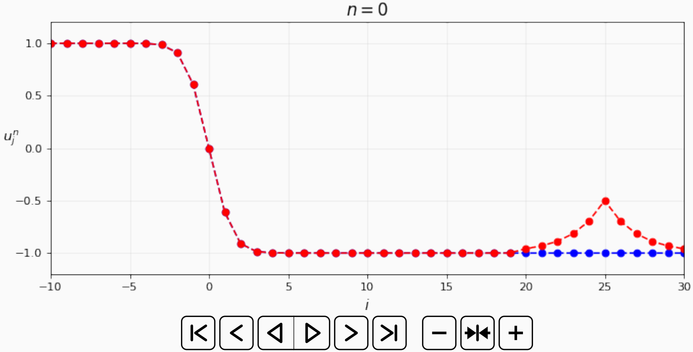
For standing Lax shocks, one aims to have the existence of a differentiable one-parameter family $(\overline{u}^{\delta})_{\delta \in]-\varepsilon,\varepsilon[}$ of SDSPs.

- Jennings, *Discrete shocks* (1974)
    - scalar case
    - conservative monotone scheme
    - for shocks satisfying Oleinik's E-condition
- Majda and Ralston, *Discrete Shock Profiles for Systems of Conservation Laws* (1979)
    - system case
    - first order scheme
    - weak Lax shocks
- Michelson, *Discrete shocks for difference approximations to systems of conservation laws* (1984)
    - extension of Majda-Ralston for third order scheme
- Different cases: Smyrlis (1990), Liu-Yu (1999), Serre (2004) etc...

Stability of discrete shock profiles

The end goal would be to prove a property of nonlinear orbital stability for the DSPs:

For small admissible perturbations h, prove that the solution u^n of the numerical scheme for the initial condition $u^0 = \overline{u} + h$ converges towards the set of translations of the DSP $\left\{\overline{u}^\delta, \delta \in]-\varepsilon, \varepsilon[\right\}$.



Known stability results

- Jennings, *Discrete shocks* (1974)
  - scalar case
  - conservative monotone scheme
  - nonlinear orbital stability for ℓ^1 perturbations
- Liu-Xin, *L^1-stability of stationary discrete shocks*, (1993)
  - system case
  - Lax-Friedrichs scheme
  - weak Lax shocks
  - zero mass perturbation (dropped in Ying (1997))
- Michelson, *Stability of discrete shocks for difference approximations to systems of conservation laws*, (2002)
  - system case
  - weak Lax shocks
  - First and third order schemes
- Different cases: Smyrlis (1990), Liu-Yu (1999), etc...

One would hope to prove a result of nonlinear orbital stabilty in the system case, for a fairly large class of numerical schemes and with no smallness assumption on the amplitude of the shock.

Our first goal is to study the semigroup $(\mathcal{L}^n)_{n\in\mathbb{N}}$ associated to the operator $\mathcal{L}$ obtained by linearizing $\mathcal{N}$ about the SDSP $\overline{u}$.

We introduce a zero mass perturbation $h^0 \in \ell^1(\mathbb{Z})$. We then define

$$v^0 = \overline{u} + h^0$$

and

$$\forall n \in \mathbb{N}, \quad v^{n+1} = \mathcal{N}(v^n). \tag{2}$$

If we define $h^n = v^n - \overline{u}$, then (2) yields

$$h^{n+1} = \mathcal{L}h^n + Q(h^n, \overline{u})$$

with $Q(h^n, \overline{u})$ being some "quadratic" term. Duhamel's formula implies that a precise understanding of the behavior of the family of operators $(\mathcal{L}^n)_{n\geq 0}$ is necessary at this point.

The second idea

Study of the Green's function associated to $\mathcal{L}$:

$$\forall j_0 \in \mathbb{Z}, \forall n \in \mathbb{N}, \quad \mathcal{G}(n, j_0, \cdot) = \mathcal{L}^n \delta_{j_0}.$$

Find a way to link spectral properties of $\mathcal{L}$ to the asymptotic behavior of the Green's function $\mathcal{G}(n, j_0, j)$

• Techniques developed in Zumbrun-Howard, *Pointwise semigroup methods and stability of viscous shock waves* (1998) to study traveling waves for parabolic PDEs.

• Extension of the result of Lafitte-Godillon, *Green's function pointwise estimates for the modified Lax-Friedrichs scheme*, (2003)

Linearization of the numerical scheme about the constant states $u^{\pm}$

We define the bounded operator $\mathcal{L}^{\pm} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ obtained by linearizing $\mathcal{N}$ about the constant state $u^{\pm}$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}^{\pm} h)_j := \sum_{k=-p}^{q} a_k^{\pm} h_{j+k}.$$

The coefficient $a_k^{\pm}$ are expressed using the partial derivatives $\partial_k F(\nu; u^{\pm}, \ldots, u^{\pm})$.

This is a Laurent operator/convolution operator. Its spectrum is given by

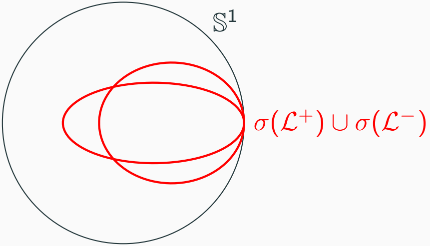
$$\sigma(\mathcal{L}^{\pm}) = \left\{ \sum_{k=-p}^{q} a_k^{\pm} e^{itk}, t \in \mathbb{R} \right\}.$$

$$\bullet \sum_{k=-p}^{q} a_k^{\pm} = 1, \quad \sum_{k=-p}^{q} k a_k^{\pm} = -\nu f'(u^{\pm}) \quad \text{(consistency condition)}$$

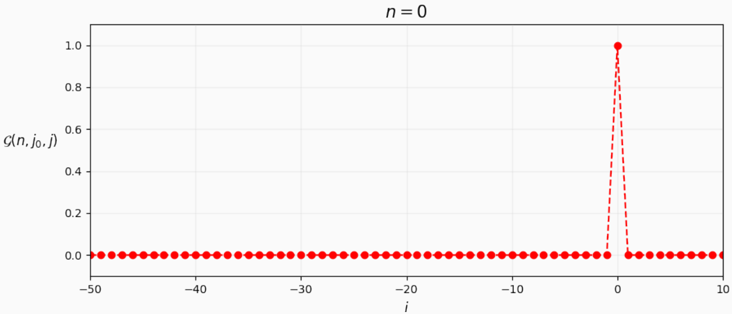
$$\bullet \forall t \in \mathbb{R} \backslash 2\pi\mathbb{Z}, \quad \left| \sum_{k=-p}^{q} a_k^{\pm} e^{itk} \right| < 1 \quad (\ell^2 - \text{stability})$$

- There exists a complex number $\beta_{\pm}$ with positive real part such that

$$\sum_{k=-p}^{q} a_k^{\pm} e^{itk} \underset{t \to 0}{=} \exp(-if'(u^{\pm})\nu t - \beta_{\pm} t^2 + O(|t|^3)). \quad \text{(Diffusivity condition)}$$



Green's function associated to the operator $\mathcal{L}^+$



The Gaussian behavior has been studied thoroughly in recent extensions on the local limit theorem (see [DSC14, RSC15, CF22, Coe22]).

Linearization of the numerical scheme about the SDSP

We define the bounded operator $\mathcal{L} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ obtained by linearizing $\mathcal{N}$ about $\overline{u}$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}h)_j := \sum_{k=-p}^{q} a_{j,k} h_{j+k},$$

with $a_{j,k} \to a_k^{\pm}$ as $j \to \pm\infty$.

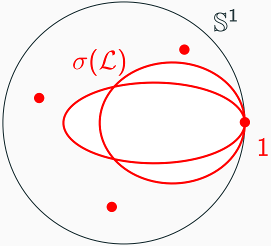
The coefficient $a_{j,k}$ are expressed using the partial derivatives $\partial_k F(\nu; \overline{u}_{j-p}, \dots, \overline{u}_{j+q-1})$.

We define the Green's function

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j \in \mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Observation on the spectrum of $\mathcal{L}$

The elements of the unbounded component of $\mathbb{C}\backslash\sigma(\mathcal{L}^+)\cup\sigma(\mathcal{L}^-)$ are either eigenvalues of $\mathcal{L}$ or are in its resolvent set.



Spectral stability assumption

- In the article, we construct a so-called Evans function. We assume that 1 is a simple zero of the Evans function. As a consequence, 1 is a simple eigenvalue of the operator $\mathcal{L}$.

$$\text{"}\mathcal{N}(\overline{u}^\delta) = \overline{u}^\delta \text{ and thus } \mathcal{L}\frac{\partial \overline{u}^\delta}{\partial \delta} = \frac{\partial \overline{u}^\delta}{\partial \delta}.\text{"}$$

- The operator $\mathcal{L}$ has no other eigenvalue of modulus equal or larger than 1.

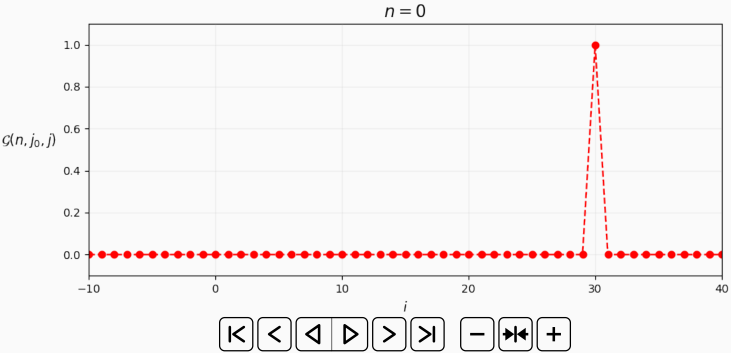
Theorem
Under some more precise assumptions, there exist a positive constant c, an element V of $\ker(Id - \mathcal{L})$ and an (explicit) function $E : \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N} \setminus \{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$
\begin{aligned}
&\mathcal{G}(n, j_0, j) \\
=&E\left(\frac{nf'(u^+)\nu + j_0}{\sqrt{n}}\right) V(j) \quad \text{(Excited eigenvector)} \\
+&\mathbb{1}_{j\in\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu - (j - j_0)|^2}{n}\right)\right)\right) \quad \text{(Gaussian wave)} \\
+&\mathbb{1}_{j\in-\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu + j_0|^2}{n}\right)\right) e^{-c|j|}\right) \quad \text{(Exponential residual)} \\
+&O(e^{-cn-c|j-j_0|})
\end{aligned}
$$

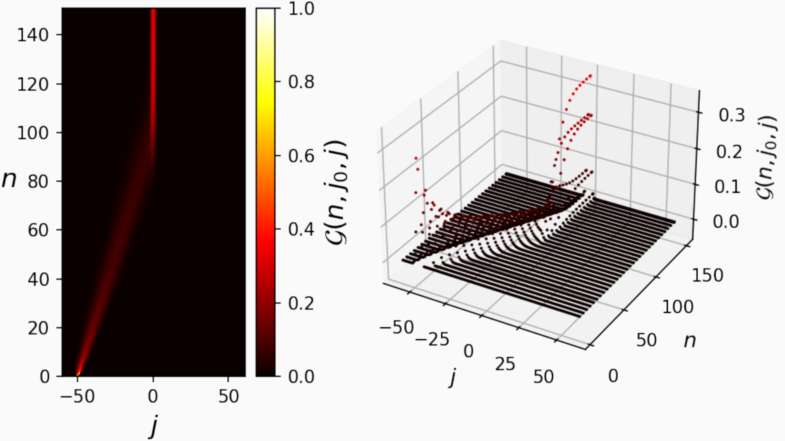
where $E(x) \underset{x\to-\infty}{\to} 1$ and $E(x) \underset{x\to+\infty}{\to} 0$.

There is a similar result for $j_0 \in -\mathbb{N}$.

Green's function associated to the operator $\mathcal{L}$ for $j_0 = 30$

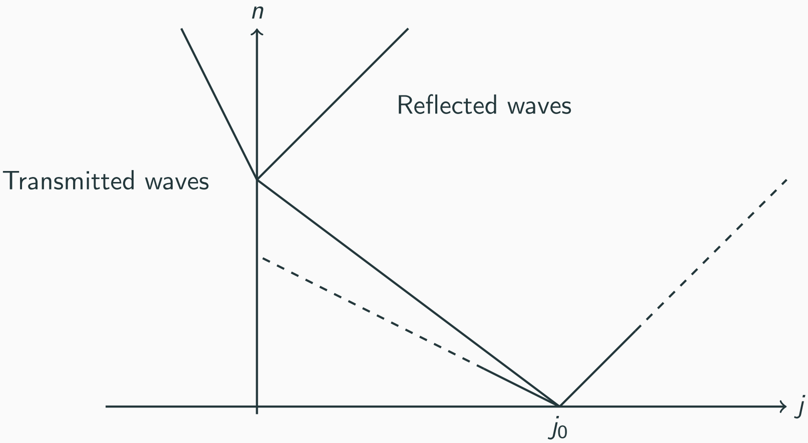


$j_0 = -50$



140
120
100
80
60
40
20
0
n
−50
0
50
j
1.0
0.8
0.6
0.4
0.2
0.0
𝒢(n, j0, j)
0.3
0.2
0.1
0.0
150
100
50
0
−50
−25
25

Case of systems



• Using the inverse Laplace tranform with Γ a path that surrounds the spectrum $\sigma(\mathcal{L})$, we have

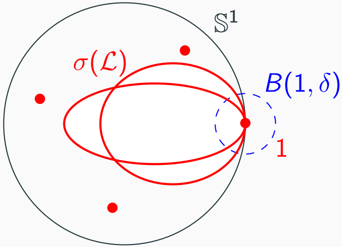
$$\forall n \in \mathbb{N}\setminus\{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi}\int_\Gamma z^n \left((zId - \mathcal{L})^{-1}\delta_{j_0}\right)_j dz. \tag{3}$$

• We rewrite the eigenvalue problem

$$(zId - \mathcal{L})u = 0$$

as a discrete dynamical system

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z)W_j. \tag{4}$$



We are interested in solutions of (4) that tend towards 0 as j tends to $+\infty$ or $-\infty$ (Jost solutions, geometric dichotomy) and use them to express find an expression and meromorphically extend $z \mapsto \left((zId - \mathcal{L})^{-1}\delta_{j_0}\right)_j$ through the essential spectrum near 1.

• Using this idea and a good choice of path Γ, we prove sharp estimates on the temporal Green's function.

Theorem

Under the same assumption as for the previous theorem, for $p \in [1, +\infty]$, there exists a positive constant C such that

$$\forall h \in \ell^1(\mathbb{Z}, \mathbb{C}^d), \forall n \in \mathbb{N}, \quad \min_{V \in \ker(Id_{\ell^2} - \mathcal{L})} \|\mathcal{L}^n h - V\|_{\ell^p} \leq \frac{C}{n^{\frac{1}{2}\left(1-\frac{1}{p}\right)}} \|h\|_{\ell^1}.$$

About the theorem:

- Bounds uniform in j_0
- Very few limitation on the size of the stencil
- The result can be proved for systems
- The result can be proved for higher odd ordered schemes (not only for first order schemes)

Perspective:

- Can we now prove nonlinear orbital stability ? (at least in the scalar case?)
- Existence of spectrally stable SDSPs?
- What can we say for moving shocks (with rational speed)?
- What can we say for dispersive schemes? (Lax-Wendroff for instance)
- Study of the stability for multi-dimensional conservation laws (Carbuncle phenomenon)

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